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# Synthesis of Essential-Regular Bicubic Impedances

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## ABSTRACT

This paper provides a complete realisation of a special class of positive-real bicubic impedances. The problem is motivated by the concept of the inerter, which is the mechanical dual of a capacitor. This device allows mechanical network synthesis, by completing the electrical mechanical analogy. With mechanical synthesis, the emphasis is to minimise the number of elements required to allow feasible implementation. The definitions of simple-series-parallel networks and essential-regular positive-real functions are introduced. The simple-series-parallel minimum reactive networks that can realise all essential-regular bicubics are identified and grouped into six network quartets. One of the advantages of these networks is that they contain the minimal number of reactive elements. The necessary and sufficient realisability conditions for all these networks, as well as corresponding element values are then derived. Finally, numerical examples are provided to illustrate the validity of the theoretical results. In the course of the argument, interesting conclusions regarding essential-regular bilinear and biquadratic functions have also been presented.

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KEY WORDS: network synthesis; bicubic admittance; essential-regular; simple-series-parallel

## 1. INTRODUCTION

Passive network synthesis is an important part of systems theory and experienced its ‘golden age’ from the 1930s to the 1970s with many elegant results published [1–6]. In 1949, Bott and Duffin [3] proposed a general transformerless synthesis procedure and using this showed that any positive real function could be realised as the driving point immittance of a network consisting of resistors, capacitors and inductors only. However, this procedure appears to be wasteful in the number of elements used. Despite the significant effort invested in resolving the minimal transformerless realisation problem, a complete picture was never obtained. Interest in this topic waned in the early 1970s due to the growing importance of integrated circuits. However, recently there has been a resurgence of interest triggered by the invention of a new mechanical network element, the inerter.

The inerter, which was introduced by Smith [7] in 2002, is a two terminal device with the property that the force at the terminals is proportional to their relative acceleration. It completes the analogy between electrical and mechanical systems. Under the force-current analogy, spring, damper and inerter correspond to inductor, resistor and capacitor, respectively. Applications of mechanical systems which contain inerters have been demonstrated for vibration isolation and suppression of road vehicles [8, 9], motorcycle steering systems [10–12], multi-storey buildings [13–15], railway vehicles [16–19] and landing gears [20]. Compared with electrical realisations,

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mechanical network synthesis has its own weight and space constraints. Hence it is crucial to minimise the number of elements required. To this end, recently, researchers have contributed to identifying the most efficient series-parallel realisation of a given impedance [21–25]. Using the Bott-Duffin procedure [3], a general biquadratic impedance requires a series-parallel network containing 6 reactive elements and 3 resistors. By defining the concept of ‘regular positive-real function’, all biquadratics that can be realised by series-parallel five-element networks [21] and six-element networks [22] have been identified, requiring three and nine network quartets respectively (the definition of network quartet is given in [21,26], and is reviewed in Section 2 of the present paper). Later on, Wang et al. [24] presented networks with four elements which can also realise some biquadratics. Three network quartets are identified in total. The realisation problem of a biquadratic function with an extra pole at the origin has been studied in [23]. An important conclusion regarding the Bott-Duffin realisation has been obtained by Hughes and Smith [25]. They showed that this procedure results in both the least number of reactive elements and resistors among all series-parallel networks realising a biquadratic minimum function.

It is apparent that impedances with higher McMillan degree, e.g. a bicubic function, can achieve a wider range of dynamic properties. For example, it has been shown in [16] that compared with using biquadratic impedances in the railway vehicle vertical suspension, bicubic functions can reduce the wheel load index by 36%. However, very limited study has been carried out on synthesizing bicubic impedances. In the only available literature, [4] and [5] back in 1970s, Tirtoprodjo focused on bicubic functions with multiple real poles and zeros, and moved on to general bicubic impedances. The realisation procedure is still highly redundant in terms of the number of reactive elements used. In the present paper, the minimum reactive synthesis (3 reactive elements for bicubics) of a special class of bicubic impedances, which we term the essential-regular bicubic impedances, have been obtained. In [16], the 36% improvement was achieved using a network with eight reactive and five resistive elements based on Bott-Duffin procedure, whereas in this paper we can achieve the same improvements using a network with three reactive and four resistive elements.

This paper is structured as follows. In Section 2, we recall some preliminary results that are needed in this study. The definitions of simple-series-parallel networks and essential-regular positive-real functions are then introduced. In Section 3, six network quartets that can realise all essential-regular positive-real bicubic functions are identified. The necessary and sufficient realisability conditions, as well as corresponding element values are also derived. Numerical examples are given in Section 4 and conclusions are drawn in Section 5.

## 2. PRELIMINARIES

In this section, we review the concept of a network quartet, regular positive-real functions and some other key conclusions reported in [21] and [26]. The concepts of a simple-series-parallel network and essential-regular positive-real functions are then introduced.

### 2.1. Relevant observations from the literature

The classification of networks is facilitated by the following transformations on the impedance  $Z(s)$  [21]:

1. Frequency inversion:  $s \rightarrow s^{-1}$ ,
2. Impedance inversion:  $Z \rightarrow Z^{-1}$ .

The first transformation corresponds to replacing inductors with capacitors of reciprocal value (and vice versa), and the second to taking the network dual. These transformations allow networks to be arranged into groups of four, which we call *network quartets* ([21], [27]) shown in Figure 1.

A *regular* positive-real function is defined as follows:

*Definition 1* ([21])

A positive-real function  $Z(s)$  is defined to be *regular* if the smallest value of  $\operatorname{Re}(Z(j\omega))$  or  $\operatorname{Re}(Z^{-1}(j\omega))$  occurs at  $\omega = 0$  or  $\omega = \infty$ .

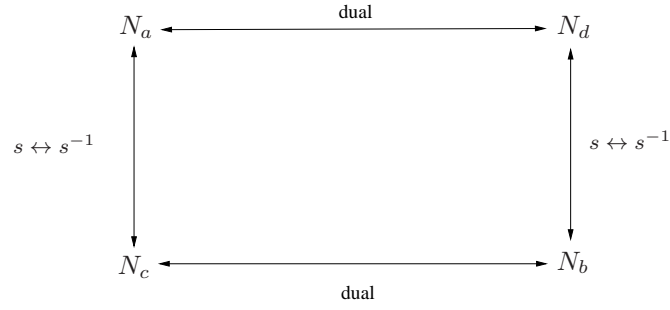


Figure 1. Transformations relating members of a network quartet.

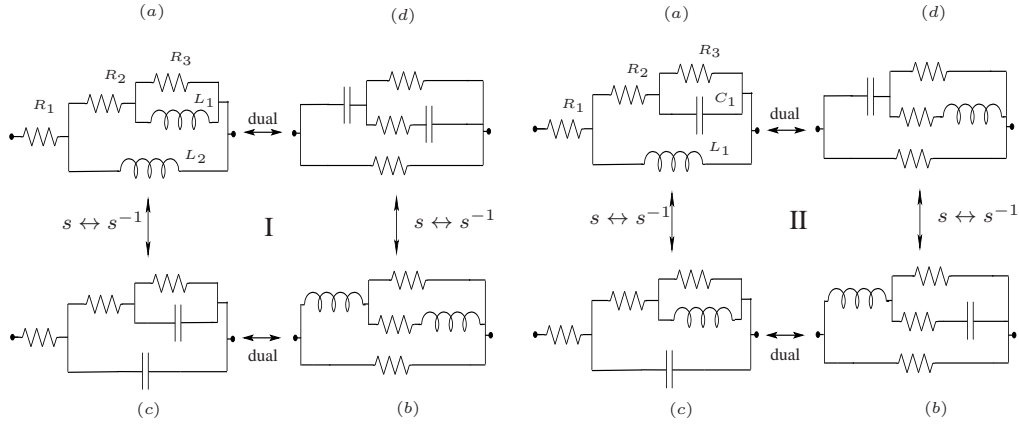


Figure 2. Two series-parallel five-element network quartet with two reactive elements that can realise all the regular biquadratic impedances (1).

A useful special property of regular positive-real functions is given in Lemma 1:

**Lemma 1** ([21])

Let  $Z(s)$  be a regular positive-real function. Then  $Z(s) + R$  and  $Z(s) + R^{-1}$  are both regular, where  $R$  is non-negative.

Now, more specially consider a biquadratic function

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}, \quad (1)$$

where  $A, B, C, D, E$ , and  $F \geq 0$ . It is well known that  $Z(s)$  is positive-real if and only if  $\sigma = BE - (\sqrt{AF} - \sqrt{CD})^2 \geq 0$  [28–30]. For the function to be regular, further conditions are placed on the coefficients as described by Lemma 2.

**Lemma 2** ([21])

A positive-real biquadratic impedance (1) is regular if and only if the conditions of at least one of the following four cases are satisfied:

Case 1.  $AF - CD \geq 0$  and  $\lambda_1 = E(BF - CE) - F(AF - CD) \geq 0$ ,

Case 2.  $AF - CD \geq 0$  and  $\lambda_2 = B(AE - BD) - A(AF - CD) \geq 0$ ,

Case 3.  $AF - CD \leq 0$  and  $\lambda_3 = D(AF - CD) - E(AE - BD) \geq 0$ ,

Case 4.  $AF - CD \leq 0$  and  $\lambda_4 = C(AF - CD) - B(BF - CE) \geq 0$ .

The authors of [26] have studied all series-parallel networks with two reactive elements and an arbitrary number of resistors. The following theorem has been obtained.

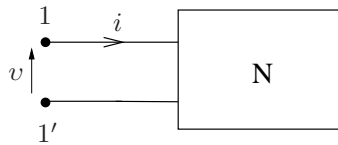


Figure 3. Electrical network  $N$  with two external terminals 1 and  $1'$ , terminal voltage  $v$  and terminal current  $i$ .

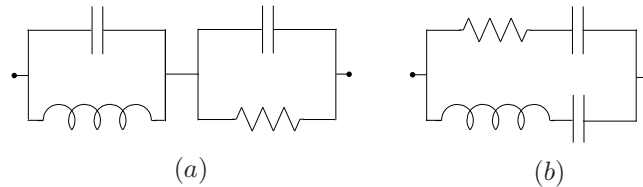


Figure 4. Networks that can only realise regular immittances based on Lemma 3.

*Theorem 1* ([26])

Any series-parallel one-port network consisting of two reactive elements and an arbitrary number of resistive elements is equivalent to a network consisting of two reactive elements and no more than three resistive elements.

A complete analysis of all five element networks with two reactive elements was then carried out in [21]. This study concluded that:

*Theorem 2* ([21])

A biquadratic impedance (1) can be realised by five-element series-parallel networks with two reactive elements if and only if it is regular. Moreover, all regular biquadratics can be realised by the network quartet of Figure 2-I or Figure 2-II. Furthermore, only two networks from the quartet of Figure 2-I (either of the networks shown in Figure 2-I(a) or (b) and either of the networks shown in Figure 2-I(c) or (d)) are needed to cover all the cases where the reactive elements are of the same kind.

Finally, the argument set out in this paper will make use of the following Lemma. For a two-terminal network (see Figure 3), a cut set is defined to be a set of branches whose removal places the two terminals 1 and  $1'$  in disconnected parts. With this definition, it can be said that:

*Lemma 3* ([21])

For any two-terminal network, it can only realise regular immittances if it contains any path or any cut set consisting only of one type of reactive element.

Using Lemma 3, the networks of Figure 4 can only realise regular immittances since network (a) has a path and (b) has a cut set consisting of capacitors only.

## 2.2. Simple-series-parallel networks and essential-regular impedances

### Definition 2

A two-terminal network is defined to be **simple-series-parallel\*** if and only if starting from a single element it can be built up via a series of steps, where each step involves connecting an additional *single* element in series or in parallel with the network (or element) obtained by the previous step.

A network containing a single element is defined to be simple-series-parallel. It can be seen that all networks shown in Figure 2-I and 2-II are simple-series-parallel. An example of non-simple-series-parallel network is shown in Figure 5.

Considering this type of network, the following important features can be identified:

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\*This definition has previously been used by R. Kalman in several talks including that of the Third Workshop on Mathematical Aspects of Network Synthesis, Würzburg, Germany in 2014

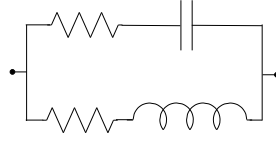


Figure 5. An example network which is not simple-series-parallel.

**Lemma 4**

Let  $Z(s)$  be an impedance of a simple-series-parallel network, then  $Z(s)$  is always regular.

**Proof**

According to Lemmas 1 and 3, the conclusion can be obtained.  $\square$

Building on this, we introduce the essential-regular functions:

**Definition 3**

Any positive and real number is **essential-regular**, and  $Z(s)$  is **essential-regular** if it is positive-real and at least one of the following four conditions holds<sup>†</sup>:

- (a) The smallest value of  $\text{Re}(Z(j\omega))$  occurs at  $\omega = 0$  and, defining  $G(s) = Z(s) - Z(0)$ , then  $H(s) = \frac{1}{G(s)} - \frac{1}{s} \lim_{s \rightarrow 0} (\frac{s}{G(s)})$  is essential regular.
- (b) The smallest value of  $\text{Re}(Z(j\omega))$  occurs at  $\omega = \infty$  and, defining  $G(s) = Z(s) - Z(\infty)$ , then  $H(s) = \frac{1}{G(s)} - s \lim_{s \rightarrow \infty} (\frac{1}{sG(s)})$  is essential regular.
- (c) The smallest value of  $\text{Re}(\frac{1}{Z(j\omega)})$  occurs at  $\omega = 0$  and, defining  $G(s) = \frac{1}{Z(s)} - \frac{1}{Z(0)}$ , then  $H(s) = \frac{1}{G(s)} - \frac{1}{s} \lim_{s \rightarrow 0} (\frac{s}{G(s)})$  is essential regular.
- (d) The smallest value of  $\text{Re}(\frac{1}{Z(j\omega)})$  occurs at  $\omega = \infty$  and, defining  $G(s) = \frac{1}{Z(s)} - \frac{1}{Z(\infty)}$ , then  $H(s) = \frac{1}{G(s)} - s \lim_{s \rightarrow \infty} (\frac{1}{sG(s)})$  is essential regular.

From Definitions 2 and 3, it can be concluded that any positive-real function with McMillan degree equal to  $n$  is essential-regular if it can be realised by simple-series-parallel networks with  $n$  reactive elements. Then, we can have the following two Lemmas.

**Lemma 5**

All bilinear functions are essential-regular.

**Proof**

For a bilinear impedance  $Z(s) = (As + B)/(Cs + D)$ , where  $A, B, C, D > 0$ , it can be verified that  $Z(s)$  is always realisable by at least one of the networks shown in Figure 6 via Foster preamble when  $AD - BC \neq 0$ . For the case  $AD - BC = 0$ ,  $Z(s)$  can be realised by a resistor. It can be checked that the networks in Figure 6 are all simple-series-parallel with one reactive element.  $\square$

**Lemma 6**

All regular biquadratic functions (1) are essential-regular.

**Proof**

It can be seen that the networks shown in Figure 2-I and 2-II are all *simple-series-parallel* with two reactive elements. The conclusion can then be drawn using Theorem 2.  $\square$

<sup>†</sup>To show that  $Z(s)$  is essential-regular, the rule must be applied iteratively until the MacMillan degree is reduce to 0 and the resulting number is shown to be positive and real.

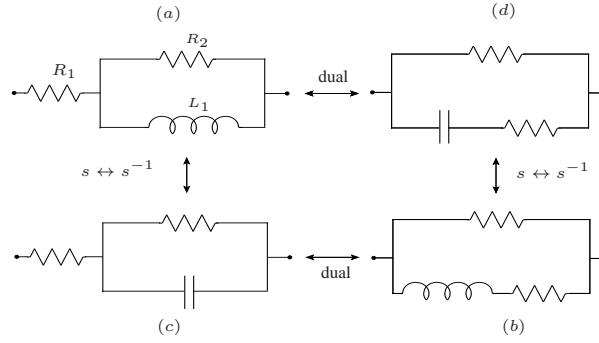


Figure 6. Series-parallel three-element network quartet that can realise all the regular bilinear impedances.

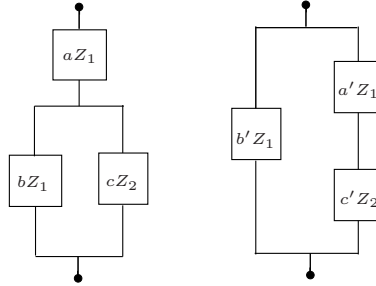


Figure 7. Two equivalent networks related by the transformations in Lemma 7.

### 3. SYNTHESIS OF ESSENTIAL-REGULAR BICUBIC FUNCTIONS

The synthesis of essential-regular bicubic functions is now considered, using the general form for the bicubic function

$$Z(s) = \frac{As^3 + Bs^2 + Cs + D}{Es^3 + Fs^2 + Gs + H} \quad (2)$$

where  $A, B, \dots, H \geq 0$ .

For any rational function  $\rho(A, \dots, H)$ , we introduce the notations  $\rho^f(A, \dots, H)$  for the frequency inversion transformation  $A \leftrightarrow D, B \leftrightarrow C, E \leftrightarrow H, F \leftrightarrow G$ ;  $\rho^I(A, \dots, H)$  for the impedance inversion transformation  $A \leftrightarrow E, B \leftrightarrow F, C \leftrightarrow G, D \leftrightarrow H$ . Both  $\rho^{fI}(A, \dots, H)$  and  $\rho^{If}(A, \dots, H)$  can represent the frequency inversion and impedance inversion combined transformation  $A \leftrightarrow H, B \leftrightarrow G, C \leftrightarrow F, D \leftrightarrow E$ .

To synthesise essential-regular bicubics, the following equivalence relationship is required:

*Lemma 7* ([31])

For arbitrary impedances  $Z_1(s), Z_2(s)$  and positive constants  $a, b, c$ , the networks shown in Figure 7 are equivalent under the transformations:  $a' = a(a+b)/b, b' = a+b, c' = c((a+b)/b)^2$ .

Using this, it can be said that:

*Theorem 3*

Any simple-series-parallel network with three reactive elements and an arbitrary number of resistive elements is equivalent to at least one of the networks shown in Figure 8.

*Proof*

We will consider a sequence of steps where at each stage a single element is connected in series or parallel to the network formed in the previous step to obtain a new simple-series-parallel network. After each step, we will carry out any obvious simplifications of the new network to obtain a simpler but equivalent two-terminal network. For example, if two resistors are connected in series or in parallel, they can be reduced again to a single resistor. Using the procedure, any simple-series-parallel network with three reactive elements can be obtained.

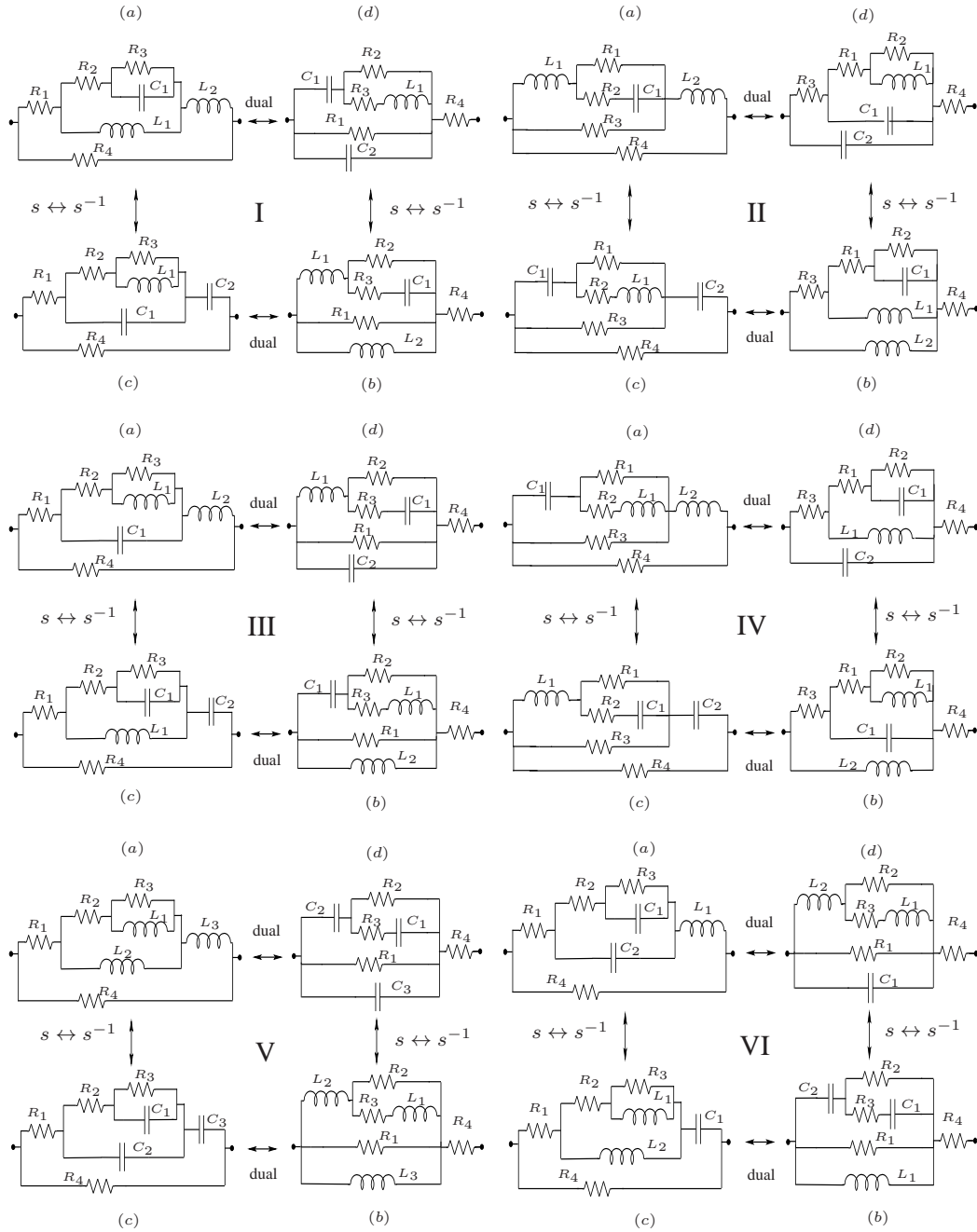


Figure 8. The six network quartets of the simple-series-parallel three-reactive-element networks that can realise all the essential-regular bicubics.

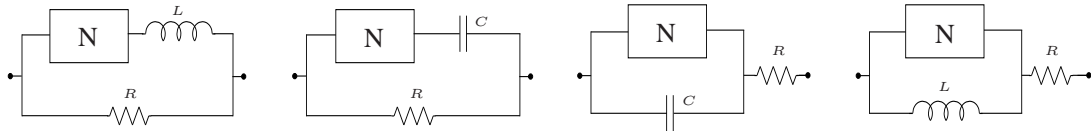


Figure 9. Networks obtained by the procedure of forming simple-series-parallel networks.

It is obvious that at some stage of the above mentioned process, a simple-series-parallel network consisting of two reactive elements (denoted as  $N$  in Figure 9) will be obtained. From Theorems 1



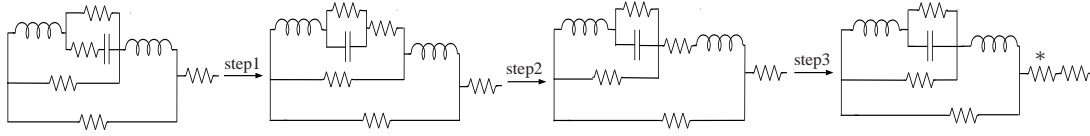


Figure 10. An example showing three applications of Lemma 7.

and 2, it can be seen that the networks shown in Figure 2-I and 2-II can cover all the cases. Using Lemma 7, it can then be checked that any subsequent step involving a series or parallel connection with a resistor will reduce down to a network with three resistors which is equivalent to the original one. So the next step must be a series or parallel connection of this sub-network  $N$  with the third reactive element and the fourth resistor can only be connected as Figure 9, correspondingly. It can then be checked using Lemma 7 that if an extra resistor is connected in series or in parallel with any of the networks shown in Figure 9, this resistor can be removed. Replacing  $N$  in Figure 9 by networks in Figure 2-I and 2-II, the six network quartets shown in Figure 8 can be obtained. Taking the network of Figure 8-II(a) as an example, adding a resistor in parallel is redundant because it can be combined with  $R_4$ . To show a resistor adding in series is redundant, Lemma 7 needs to be applied three times. This is illustrated in Figure 10, where after Lemma 7 is applied three times, it can be seen that the resistor (marked with an asterisk) can be removed.  $\square$

Based on Theorem 3, all the essential-regular bicubic functions (2) can be realised by the networks shown in Figure 8. Next, we define necessary and sufficient conditions for  $Z(s)$  in (2) to be realised by each of the six quartets. For each quartet, only one network needs to be investigated, since the realisability conditions of the remaining networks can be immediately obtained by the transformation for dual and frequency-inversed networks.

#### Theorem 4

A bicubic impedance  $Z(s)$  (2), with  $A, \dots, H > 0$  can be realised by the network shown in Figure 8-I(a) with  $L_1, L_2, C_1$  positive and finite if and only if

$$a_1 > 0, a_2 \geq 0, a_3 > 0, a_4 \geq 0, a_5 \geq 0, \quad (3)$$

$$\eta \geq 0, \zeta_1 \geq 0, \varepsilon > 0, a_3a_5 - Da_1a_2 > 0. \quad (4)$$

where

$$\begin{aligned} a_1 &= AF - BE, a_2 = AG - CE, a_3 = AH - ED, \\ a_4 &= B(AF - BE) - A(AG - CE), a_5 = C(AF - BE) - A(AH - ED), \\ \eta &= a_3a_4 - Da_1^2, \zeta_1 = a_2(a_3a_5 - Da_1a_2) - a_3\eta, \varepsilon = \eta^2 - (a_2a_4 - a_1a_5)(a_3a_5 - Da_1a_2). \end{aligned} \quad (5)$$

Furthermore, if conditions (3), (4) are satisfied, the element values can be expressed as

$$\begin{aligned} R_1 &= \frac{AD}{a_3}, R_2 = \frac{\eta A}{a_1^2 a_3}, R_3 = \frac{\varepsilon A}{a_1^2 \zeta_1}, R_4 = \frac{A}{E}, \\ L_1 &= \frac{(a_3a_5 - Da_1a_2)A}{a_1a_3^2}, C_1 = \frac{(a_3a_5 - Da_1a_2)a_1^3}{A\varepsilon}, L_2 = \frac{A^2}{a_1}. \end{aligned} \quad (6)$$

#### Proof

**Necessity.** For the network shown in Figure 8-I(a), the admittance can be expressed as

$$Y(s) = \frac{1}{Z_1(s) + L_2s} + \frac{1}{R_4} \quad (7)$$

where  $L_2$  is positive and finite and

$$Z_1(s) = \frac{A_1s^2 + B_1s + C_1}{D_1s^2 + E_1s + F_1}, \quad (8)$$

Table I. Necessary and sufficient conditions for the bicubics (2) to be realised by the networks shown in Figure 8-(I-VI)(a).

Networks	Necessary and sufficient realizability conditions	
Figure 8-I (a)	$a_1 > 0,$ $a_2 \geq 0, a_3 \geq 0,$ $a_4 \geq 0, a_5 \geq 0$	$\eta \geq 0, \varepsilon > 0, \zeta_1 \geq 0, a_3 a_5 - D a_1 a_2 > 0, a_3 \neq 0$
Figure 8-II (a)		$\eta \geq 0, \varepsilon > 0, \zeta_2 \geq 0, a_2 a_4 - a_1 a_5 > 0, a_4 \neq 0$
Figure 8-III (a)		$\eta \leq 0, \varepsilon > 0, \zeta_3 \geq 0, a_2 a_4 - a_1 a_5 < 0, a_3 \neq 0$
Figure 8-IV (a)		$\eta \leq 0, \varepsilon > 0, \zeta_4 \geq 0, a_3 a_5 - D a_1 a_2 < 0, a_4 \neq 0$
Figure 8-V (a)		$\eta \geq 0, \varepsilon < 0, a_3 a_5 - D a_1 a_2 > 0, a_3 \neq 0$
Figure 8-VI (a)		$\eta \leq 0, \varepsilon < 0, a_2 a_4 - a_1 a_5 < 0$

which is the impedance of the network shown in Figure 2-II(a). It can be seen that for the case where  $R_4$  equals 0 or  $\infty$ , the impedance  $Z(s)$  will be zero or  $E = 0$ , respectively. Hence, we assume  $R_4$  is positive and finite.

Using (7), it can be easily obtained that

$$R_4 = 1 / \lim_{s \rightarrow \infty} (1 / (Z_1(s) + L_2 s) + 1 / R_4) = 1 / \lim_{s \rightarrow \infty} (Y(s)) = A / E, \quad (9)$$

and

$$L_2 = \lim_{s \rightarrow \infty} (Z_1 / s + L_2) = \lim_{s \rightarrow \infty} (1 / (s(Y(s) - E / A))) = A^2 / a_1 \quad (10)$$

where  $a_1$  is positive and shown in equation (5).

First, substituting (8), (9), (10) into (7) and equating (2) with the inverse of (7), then expressing  $A_1, B_1, \dots, F_1$  with parameters  $A, B, \dots, H$ , we can obtain

$$A_1 = \frac{k E a_4}{A a_1}, B_1 = \frac{k E a_5}{A a_1}, C_1 = \frac{k D E}{A}, D_1 = \frac{k E a_1}{A^2}, E_1 = \frac{k E a_2}{A^2}, F_1 = \frac{k E a_3}{A^2}.$$

where  $k$  is a positive constant and  $a_1, \dots, a_5$  can be found in (5). For  $Z_1(s)$  to be realisable by the network of Figure 2-II(a) with  $L_1, C_1$  positive and finite, based on [20, Table 1 and Theorem 10], the parameters  $A_1, \dots, F_1$  should be non-negative and satisfy

$$\begin{aligned} A_1 F_1 - C_1 D_1 &\geq 0, \\ \lambda_1 = E_1 (B_1 F_1 - C_1 E_1) - F_1 (A_1 F_1 - C_1 D_1) &\geq 0, \\ K = (A_1 F_1 - C_1 D_1)^2 - (A_1 E_1 - B_1 D_1)(B_1 F_1 - C_1 E_1) &> 0, \\ D_1 > 0, F_1 > 0, B_1 F_1 - C_1 E_1 &> 0. \end{aligned} \quad (11)$$

For  $A_1, \dots, F_1$  to be non-negative and  $D_1 > 0, F_1 > 0$ , condition (3) can be obtained. By substituting  $A_1, \dots, F_1$  to (11), condition (4) can then be obtained, respectively. Furthermore, according to the element values derived in [20, Theorem 10], we obtain  $R_1, R_2, R_3, L_1$  and  $C_1$  shown in (6).

**Sufficiency.** Given a positive-real bicubic impedance (2) with  $A, B, C, D, E, F, G, H > 0$  satisfying the conditions (3) and (4), we can calculate the values of  $R_1, R_2, R_3, R_4, C_1, L_1$  and  $L_2$ , according to (6). Then we can calculate the impedance of the network of Figure 8-I(a) and we find that it equals the given bicubic function.  $\square$

Based on a similar procedure, we can obtain the necessary and sufficient realisability conditions for the networks shown in Figure 8-(II-VI)(a). The results are summarised in Table I. The element values of these networks are provided in the Appendix, Table II, where  $a_1, \dots, a_5, \eta$  and  $\varepsilon$  are

shown in (5) and

$$\begin{aligned}\zeta_2 &= a_5(a_2a_4 - a_1a_5) - a_4\eta, \\ \zeta_3 &= a_1\eta - a_2(a_2a_4 - a_1a_5), \\ \zeta_4 &= Da_1\eta - a_5(a_3a_5 - Da_1a_2).\end{aligned}$$

The realisability conditions of the remaining networks shown in Figure 8 can be directly obtained from the conditions shown in Table I by using the transformations  $\rho^{If}$ ,  $\rho^f$  and  $\rho^I$  for the (b), (c) and (d) networks in a network quartet, respectively. As an example, the realisability conditions for the network quartet shown in Figure 8-I have been given in the Appendix, Table III. Furthermore, the element values of the networks of Figure 8-(I-VI)(a) have been summarised in Table II in the Appendix. The element values of the remaining networks in Figure 8 can be calculated directly based on Table II using transformations shown in Table IV.

#### 4. NUMERICAL EXAMPLES

##### 4.1. Case studies of synthesising essential-regular bicubics

Using the method proposed in this paper, there are two steps to realise a given bicubic function. First, we need to identify whether it is essential-regular based on the realisability conditions of the networks shown in Figure 8. The realisability conditions for the networks shown in Figure 8-(I-VI)(a) can be found in Table I. The realisability conditions for the networks of Figure 8-I are summarised in Table III. For the rest of the networks in Figure 8, the realisability conditions can be directly obtained by following the procedure discussed at the end of Section 3. Second, if it is essential-regular, the realisable network and the corresponding element values can be obtained using Table II and IV, shown in the Appendix. To show the application of the synthesis approach, three numerical examples are provided.

The first and second examples are taken from [16], where the third order transfer functions  $K_1^{3rd}(s)$  and  $K_2^{3rd}(s)$  were shown to reduce the wheel load index of a train suspension system by 36% comparing with biquadratic functions. This type of improvement is one of the motivations for developing the minimal realisation method presented here. The expressions of the two bicubic functions are listed below,

$$\begin{aligned}K_1^{3rd}(s) &= \frac{113.45s^3 + 4.3781 \times 10^5 s^2 + 9981.5s + 2.3395e^{-5}}{s^3 + 5.0512 \times 10^3 s^2 + 5.2554 \times 10^5 s + 165.5255}, \\ K_2^{3rd}(s) &= \frac{3.2132 \times 10^{12} s^3 + 5.8967 \times 10^{18} s^2 + 5.5411 \times 10^9 s + 5.4456 \times 10^{-3}}{s^3 + 5.9915 \times 10^{12} s^2 + 160.1536s + 6.4593 \times 10^{-10}}.\end{aligned}$$

We note that these transfer functions, which result from the optimisation of a cost function using the LMI method, have a wide spread of coefficient values. The authors of [16] followed the Bott-Duffin procedure to realise these two transfer functions, resulting in eight reactive and five resistive mechanical elements being required. The two network realisations for  $K_1^{3rd}$  and  $K_2^{3rd}$  have been shown in Figure 11(a) and Figure 12(a) respectively, with corresponding element values ( $k_i$  in N/m,  $c_i$  in Ns/m,  $b_i$  in kg). Note that there is a significant spread in these element values, as can be expected given the spread of coefficient values in  $K_1^{3rd}$  and  $K_2^{3rd}$ . We now show that using the approaches proposed in the present paper, both functions are realisable using three reactive and four resistive elements. Again, the element values have a large spread, and so we include a third example at the end of this section which has a much smaller spread in element values.

For  $K_1^{3rd}(s)$ , it can be calculated that  $a_1 = 1.35 \times 10^5$ ,  $a_2 = 5.96 \times 10^7$ ,  $a_3 = 1.88 \times 10^4$ ,  $a_4 = 5.25 \times 10^{10}$  and  $a_5 = 1.35 \times 10^9$  using equations in (5). Based on Table I, it might be realised by one of the networks of Figure 8-(I-VI)(a). It can be further calculated that  $\eta = 9.85 \times 10^{14}$ ,  $\varepsilon = -7.81 \times 10^{31}$  and  $a_3a_5 - Da_1a_2 = 2.53 \times 10^{13}$ . Hence, it can be realised by the network of Figure 8-V(a) whose element values can be calculated using Table II in the Appendix as  $R_1 = 1.41 \times 10^{-7}\Omega$ ,  $R_2 = 0.019\Omega$ ,  $R_3 = 325.28\Omega$ ,  $R_4 = 113.45\Omega$ ,  $L_1 =$

$0.747H$ ,  $L_2 = 60.2H$ , and  $L_3 = 0.095H$ . Using the force-current mechanical-electrical analogy [7], the mechanical realisation of  $K_1^{3rd}(s)$  shown in Figure 11(b) can be obtained with corresponding element values ( $k_i$  in N/m,  $c_i$  in Ns/m,  $b_i$  in kg).

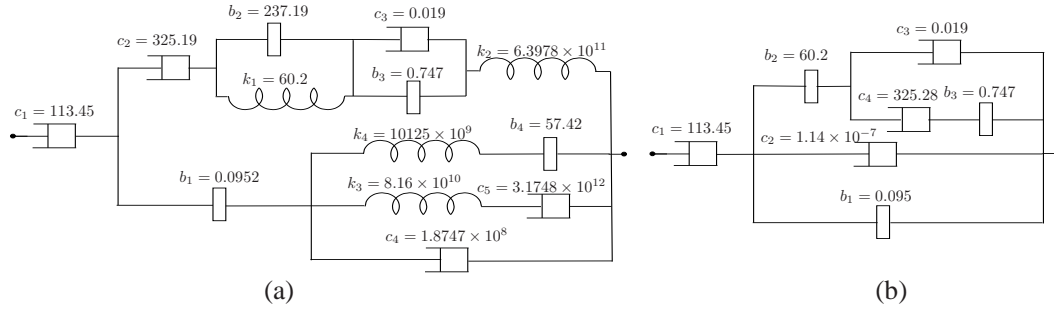


Figure 11. The network realisations for  $K_1^{3rd}$  via: (a) the Bott-Duffin procedure, (b) the Essential-regular synthesis method.

For  $K_2^{3rd}(s)$ , it can be calculated that  $a_1 = 1.93 \times 10^{25}$ ,  $a_2 = 5.15 \times 10^{14}$ ,  $a_3 = 2.08 \times 10^3$ ,  $a_4 = 1.14 \times 10^{44}$  and  $a_5 = 1.07 \times 10^{35}$  using equations in (5). Based on Table I, it can be further calculated that  $\eta = -1.78 \times 10^{48}$ ,  $\varepsilon = 3.37 \times 10^{98}$ ,  $\zeta_3 = 9.92 \times 10^{74}$  and  $a_2 a_4 - a_1 a_5 = -1.99 \times 10^{60}$ . Hence, it can be realised by the network of Figure 8-III(a) and the element values can be obtained as  $R_1 = 9.84 \times 10^5 \Omega$ ,  $R_2 = 7.45 \times 10^6 \Omega$ ,  $R_3 = 2.73 \times 10^7 \Omega$ ,  $R_4 = 3.21 \times 10^{12} \Omega$ ,  $L_1 = 6.55 \times 10^{18} H$ ,  $L_2 = 0.536H$ , and  $C_1 = 1.11 \times 10^3 F$  using Table II. The mechanical realisation of it has been obtained in Figure 12(b), where corresponding parameter values ( $k_i$  in N/m,  $c_i$  in Ns/m,  $b_i$  in kg) have also been provided.

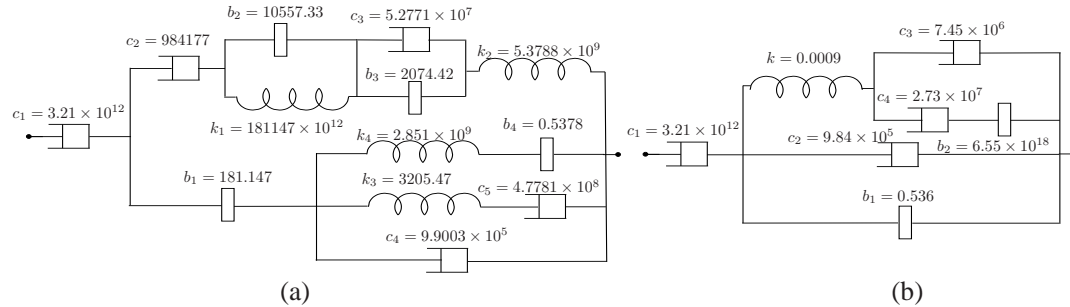


Figure 12. The network realisations for  $K_2^{3rd}$  via: (a) the Bott-Duffin procedure, (b) the Essential-regular synthesis method.

The advantage of the new procedure is that rather than the 13 component networks identified using Bott-Duffin procedure, here the transfer functions are exactly realised with just 7 elements. Further simplification may be possible by considering the significance of the various components, however this will lead to the transfer function only being approximately realised. The acceptability of such an approximation is application specific so is not considered here, see [15, 20] for examples of such approximations.

To further illustrate the validity of the proposed synthesis method, an additional example in which there is a much smaller spread of parameter values in the transfer function is now given. Taking a bicubic function with  $A = 30$ ,  $B = 630$ ,  $C = 2900$ ,  $D = 2000$ ,  $E = 21$ ,  $F = 405$ ,  $G = 1650$  and  $H = 1000$ , it can be calculated that  $a_1^f > 0$ ,  $a_2^f > 0$ ,  $a_3^f > 0$ ,  $a_4^f > 0$  and  $a_5^f > 0$ , hence it can be realised by one of the networks shown in Figure 8-(I-VI)(c). Then by checking that  $\eta^f > 0$  and  $\varepsilon^f < 0$ , this bicubic function can be realised by the network of Figure 8-V(c). According to Tables II and IV, the element values can be obtained as  $R_1 = 5\Omega$ ,  $R_2 = 3\Omega$ ,  $R_3 = 2\Omega$ ,  $R_4 = 2\Omega$ ,  $C_1 = 0.1F$ ,  $C_2 = 0.05F$ , and  $C_3 = 0.1F$ .

#### 4.2. Comparison between regular and essential-regular bicubics

In this sub-section, a remark on the relation between regular and essential-regular bicubics will be drawn. It has been shown in Lemmas 5 and 6 that regular bilinear or biquadratic functions are all essential-regular. For bicubic functions, we take an example with  $A = 1, B = 2, C = 4, D = 1, E = 1, F = 3, G = 6.5$  and  $H = 2$ . It can be checked that this bicubic function is not essential-regular using Tables I, III and transformations  $\rho^I, \rho^f, \rho^{If}$ . Then based on Lemma 8 in Appendix B, it can be calculated that the given coefficients satisfy the regularity conditions in Case 1. This highlights the following fact:

*Remark 1*

In contrast to bilinear and biquadratic functions (Lemma 5 and Lemma 6 respectively), the simple-series-parallel networks with three reactive elements cannot realize all the regular bicubics. In other words, essential-regular conditions are sufficient but not necessary for a bicubic function to be regular.

### 5. CONCLUSION

This paper constitutes a step towards the complete minimum realisation problem of bicubic impedances, which, in previous studies, have been shown to provide dynamic performance advantages for mechanical structures. A special class of bicubics, defined as essential-regular, has been analyzed. It has been shown that six minimum-reactive series-parallel network quartets are sufficient to cover all essential-regular bicubics. Necessary and sufficient conditions have been derived for a bicubic function to be realised by these networks. Furthermore, explicit expressions for the element values in these networks are provided. Finally, three numerical examples have been presented to illustrate the process of first establishing if the bicubic is essential-regular and then, if the condition is met, identifying the suitable network configuration and component values.

### APPENDIX A

The element values of the networks shown in Figure 8-(I-VI)(a) have been obtained and summarised in Table II.

To demonstrate the transformation of realisability conditions within a network quartet, the necessary and sufficient conditions for the four networks in the quartet of Figure 8-I have been shown in Table III. The realisability conditions for the networks in the other quartets of Figure 8 can be derived based on Table I using the same transformations.

Based on the element values shown in Table II for the networks of Figure 8-(I-VI)(a), the element values of the other networks can be derived directly using the transformations shown in Table IV.

### APPENDIX B

*Lemma 8*

A bicubic function  $Z(s)$  in the form of (2), where  $A, B, \dots, H > 0$ , is regular if and only if at least one of the following four cases are satisfied:

Case 1.  $\sigma_0 \geq 0, \sigma_1 \geq 0, \sigma_2 \geq 0, \sigma_3 \geq 0, \sigma_4 \geq 0$ ,

Case 2.  $\sigma_0^{If} \geq 0, \sigma_1^{If} \geq 0, \sigma_2^{If} \geq 0, \sigma_3^{If} \geq 0, \sigma_4^{If} \geq 0$ ,

Case 3.  $\sigma_0^f \geq 0, \sigma_1^f \geq 0, \sigma_2^f \geq 0, \sigma_3^f \geq 0, \sigma_4^f \geq 0$ ,

Case 4.  $\sigma_0^I \geq 0, \sigma_1^I \geq 0, \sigma_2^I \geq 0, \sigma_3^I \geq 0, \sigma_4^I \geq 0$ ,

Table II. The element values of the networks shown in Figure 8-(I-VI)(a).

Networks	The values of the elements
Figure 8-I(a)	$R_1 = \frac{AD}{a_3}, R_2 = \frac{\eta A}{a_1^2 a_3}, R_3 = \frac{\varepsilon A}{a_1^2 \zeta_1}, R_4 = \frac{A}{E}$ $L_1 = \frac{(a_3 a_5 - D a_1 a_2) A}{a_1 a_3^2}, L_2 = \frac{A^2}{a_1}, C_1 = \frac{(a_3 a_5 - D a_1 a_2) a_1^3}{A \varepsilon}.$
Figure 8-II(a)	$R_1 = \frac{a_4 AD}{\eta}, R_2 = \frac{AD \zeta_2}{\varepsilon}, R_3 = \frac{A a_4}{a_1^2}, R_4 = \frac{A}{E}$ $L_1 = \frac{A a_4^2}{a_1 (a_2 a_4 - a_1 a_5)}, L_2 = \frac{A^2}{a_1}, C_1 = \frac{\varepsilon}{AD^2 a_1 (a_2 a_4 - a_1 a_5)}.$
Figure 8-III(a)	$R_1 = \frac{a_4 A}{a_1^2}, R_2 = -\frac{\eta A}{a_3 a_1^2}, R_3 = \frac{\varepsilon A}{a_1 a_3 \zeta_3}, R_4 = \frac{A}{E}$ $L_1 = \frac{\varepsilon A}{a_1 a_3^2 (a_1 a_5 - a_2 a_4)}, L_2 = \frac{A^2}{a_1}, C_1 = \frac{a_1^3}{A (a_1 a_5 - a_2 a_4)}.$
Figure 8-IV(a)	$R_1 = -\frac{a_4 AD}{\eta}, R_2 = \frac{a_4 A \zeta_4}{a_1 \varepsilon}, R_3 = \frac{AD}{a_3}, R_4 = \frac{A}{E}$ $L_1 = \frac{a_4^2 A (a_1 a_2 D - a_3 a_5)}{a_1 \varepsilon}, L_2 = \frac{A^2}{a_1}, C_1 = \frac{a_1 a_2 D - a_3 a_5}{a_1 AD^2}.$
Figure 8-V(a)	$R_1 = \frac{AD}{a_3}, R_2 = \frac{(a_1 a_2 D - a_3 a_5)^2 A}{a_1 a_3 \zeta_1}, R_3 = -\frac{\varepsilon A}{a_1^2 \zeta_1}, R_4 = \frac{A}{E}$ $L_1 = \frac{\varepsilon A (a_1 a_2 D - a_3 a_5)}{a_1 \zeta_1^2}, L_2 = \frac{(a_3 a_5 - a_1 a_2 D) A}{a_1 a_3^2}, L_3 = \frac{A^2}{a_1}.$
Figure 8-VI(a)	$R_1 = \frac{a_4 A}{a_1^2}, R_2 = \frac{(a_1 a_5 - a_2 a_4)^2 A}{a_1^2 \zeta_3}, R_3 = -\frac{\varepsilon A}{a_1 a_3 \zeta_3}, R_4 = \frac{A}{E}$ $C_1 = \frac{\zeta_3^2 a_1}{A (a_2 a_4 - a_1 a_5) \varepsilon}, C_2 = \frac{a_1^3}{A (a_1 a_5 - a_2 a_4)}, L_1 = \frac{A^2}{a_1}.$

Table III. Necessary and sufficient conditions for the bicubics (2) to be realised by the network quartet shown in Figure 8-I.

Networks	Necessary and sufficient conditions
Figure 8-I(a)	$a_1 > 0, a_2 \geq 0, a_3 > 0, a_4 \geq 0, a_5 \geq 0$ $\eta \geq 0, \varepsilon > 0, \zeta_1 \geq 0, a_3 a_5 - D a_1 a_2 > 0.$
Figure 8-I(b)	$a_1^{If} > 0, a_2^{If} \geq 0, a_3^{If} > 0, a_4^{If} \geq 0, a_5^{If} \geq 0$ $\eta^{If} \geq 0, \varepsilon^{If} > 0, \zeta_1^{If} \geq 0, (a_3 a_5 - D a_1 a_2)^{If} > 0.$
Figure 8-I(c)	$a_1^f > 0, a_2^f \geq 0, a_3^f > 0, a_4^f \geq 0, a_5^f \geq 0$ $\eta^f \geq 0, \varepsilon^f > 0, \zeta_1^f \geq 0, (a_3 a_5 - D a_1 a_2)^f > 0.$
Figure 8-I(d)	$a_1^I > 0, a_2^I \geq 0, a_3^I > 0, a_4^I \geq 0, a_5^I \geq 0$ $\eta^I \geq 0, \varepsilon^I > 0, \zeta_1^I \geq 0, (a_3 a_5 - D a_1 a_2)^I > 0.$

where

$$\begin{aligned}
\sigma_0 &= F(CH - GD) - H(AH - ED), \\
\sigma_1 &= G(CH - GD) - H(BH - DF), \\
\sigma_2 &= AH - ED, \\
\sigma_3 &= BH - DF, \\
\sigma_4 &= \sigma_0 \sigma_3 - ((CH - DG)\sqrt{E} - \sqrt{\sigma_1 \sigma_2})^2
\end{aligned} \tag{12}$$

*Proof*

Firstly, let us assume  $Re(Z(j\omega))$  achieves its minimum when  $\omega = 0$ . This is equivalent to  $Z(s) -$

Table IV. The element values of the networks shown in Figure 8-(I-VI) (b), (c), (d).

Networks	The values of the elements
Figure 8-(I-VI)(b)	$R_1^b = \frac{1}{(R_1^a)^{If}}, R_2^b = \frac{1}{(R_2^a)^{If}}, R_3^b = \frac{1}{(R_3^a)^{If}}, R_4^b = \frac{1}{(R_4^a)^{If}}$ $L_1^b = \frac{1}{(L_1^a)^{If}}, L_2^b = \frac{1}{(L_2^a)^{If}}, L_3^b = \frac{1}{(L_3^a)^{If}}, C_1^b = \frac{1}{(C_1^a)^{If}}, C_2^b = \frac{1}{(C_2^a)^{If}}.$
Figure 8-(I-VI)(c)	$R_1^c = (R_1^a)^f, R_2^c = (R_2^a)^f, R_3^c = (R_3^a)^f, R_4^c = (R_4^a)^f$ $L_1^c = \frac{1}{(C_1^a)^f}, L_2^c = \frac{1}{(C_2^a)^f}, C_1^c = \frac{1}{(L_1^a)^f}, C_2^c = \frac{1}{(L_2^a)^f}, C_3^c = \frac{1}{(L_3^a)^f}.$
Figure 8-(I-VI)(d)	$R_1^d = \frac{1}{(R_1^a)^I}, R_2^d = \frac{1}{(R_2^a)^I}, R_3^d = \frac{1}{(R_3^a)^I}, R_4^d = \frac{1}{(R_4^a)^I}$ $L_1^d = (C_1^a)^I, L_2^d = (C_2^a)^I, C_1^d = (L_1^a)^I, C_2^d = (L_2^a)^I, C_3^d = (L_3^a)^I.$

$Z(0)$  being positive-real, hence subtracting  $Z(0)$  from  $Z(s)$ , we can obtain

$$Z_1(s) = Z(s) - Z(0) = \frac{(AH - ED)s^3 + (BH - FD)s^2 + (CH - GD)s}{EHs^3 + FHS^2 + GHS + H^2}$$

Non-negative coefficients can be obtained under the conditions  $AH - ED \geq 0$ ,  $BH - DF \geq 0$ ,  $CH - GD \geq 0$ . It can then be calculated that

$$Z_2(s) = 1/Z_1(s) - H^2/((CH - GD)s), \quad (13)$$

where  $Z_2(s) = (A_1s^2 + B_1s + C_1)/(D_1s^2 + E_1s + F_1)$  and  $A_1 = EH(CH - DG)$ ,  $B_1 = H\sigma_0$ ,  $C_1 = H\sigma_1$ ,  $D_1 = (CH - DG)\sigma_2$ ,  $E_1 = (CH - GD)\sigma_3$ ,  $F_1 = (CH - DG)^2$ . From (13), it can be seen that  $Z_1(s)$  is positive-real if and only if  $Z_2(s)$  is positive-real. For  $Z_2(s)$  being positive-real,  $A_1, B_1, \dots, F_1$  should be nonnegative and satisfy the condition that  $B_1E_1 - (\sqrt{A_1F_1} - \sqrt{C_1D_1})^2 \geq 0$ . For  $A_1, B_1, \dots, F_1$  to be nonnegative, we can obtain  $CH - DG \geq 0$ ,  $\sigma_0 \geq 0$ ,  $\sigma_1 \geq 0$ ,  $\sigma_2 \geq 0$ ,  $\sigma_3 \geq 0$ . It can then be checked that  $\sigma_0 \geq 0$  and  $\sigma_2 \geq 0$  imply  $CH - DG \geq 0$ , hence the condition  $\sigma_0 \geq 0$ ,  $\sigma_1 \geq 0$ ,  $\sigma_2 \geq 0$ ,  $\sigma_3 \geq 0$  can be obtained. For the condition  $B_1E_1 - (\sqrt{A_1F_1} - \sqrt{C_1D_1})^2 \geq 0$ , it can be obtained  $\sigma_4 \geq 0$ . Similar arguments hold for the other three cases.  $\square$

## REFERENCES

1. Brune O. Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency. *Journal of Mathematical Physics* 1931; **10**:191–236.
2. Kim HK. Transformerless zero removal synthesis. *Journal of Franklin Institute* 1966; **282**:242–259.
3. Bott R, Duffin RJ. Impedance synthesis without use of transformers. *Journal of Applied Physics* 1949; **20**:816.
4. Tirtoprodjo S. Bicubic functions with real multiple poles and zeros. *IEEE Transactions on Circuit Theory* July 1971; **18**:470–471.
5. Tirtoprodjo S. Series-parallel RLC synthesis of bicubic impedance. *Electronics letters* August 1973; **9**:370–372.
6. Anderson BDO, Moylan PJ. The Brune synthesis in state-space terms. *International Journal of Circuit Theory and Applications* 1975; **3**(2):193–199.
7. Smith MC. Synthesis of mechanical networks: the inerter. *IEEE Trans. Automatic Control* 2002; **47**(10):1648–1662.
8. Smith MC, Wang FC. Performance benefits in passive vehicle suspensions employing inerters. *Vehicle System Dynamics* 2004; **42**:235–237.
9. Scheibe F, Smith MC. Analytical solutions for optimal ride comfort and tyre grip for passive vehicle suspensions. *Vehicle System Dynamics* 2009; **47**(10):1229–1252.
10. Evangelou S, Limebeer DJN, Sharp RS, Smith MC. Mechanical steering compensation for high-performance motorcycles. *Transactions of ASME, Journal of Applied Mechanics* 2007; **74**:332–346.
11. Evangelou S, Limebeer DJN, Sharp RS, Smith MC. Control of motorcycle steering instabilities - passive mechanical compensators incorporating inerters. *IEEE Control Systems Magazine* October 2006; **26**(5):78–88.



12. Jiang JZ, Smith MC, Houghton NE. Experimental testing and modelling of a mechanical steering compensator. *The 3rd International Symposium on Communications, Control and Signal Processing (ISCCSP)*, 2007; 249–254.
13. Wang FC, Su WJ, Chen CW. Building suspensions with inerters. *Proceedings of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science* 2010; **224**:1650–1616.
14. Lazar IF, Neild S, Wagg D. Using an inerter-based device for structural vibration suppression. *Earthquake Engineering and Structural Dynamics* November 2014; **43**:1129–1147.
15. Zhang SY, Jiang JZ, Neild S. Optimal configurations for a linear vibration suppression device in a multi-storey building. *Structural Control and Health Monitoring* 2016; DOI: **10.1002/stc.1887**.
16. Wang FC, Liao MK, Liao BH, Su WJ, Chan H. The performance improvements of train suspension systems with mechanical networks employing inerters. *Vehicle System Dynamics: International Journal of Vehicle Mechanics and Mobility* 2009; **47**:805–830.
17. Wang FC, Hsieh MR, Chen HJ. Stability and performance analysis of a full-train system with inerters. *Vehicle System Dynamics* 2011; **50**(4):545–571.
18. Jiang JZ, Matamoros-Sanchez AZ, Goodall RM, Smith MC. Passive suspensions incorporating inerters for railway vehicles. *Vehicle System Dynamics: International Journal of Vehicle Mechanics and Mobility* August 2011; **50**(Sup1):263–276.
19. Jiang JZ, Matamoros-Sanchez AZ, Zolotas A, Goodall R, Smith M. Passive suspensions for ride quality improvement of two-axle railway vehicles. *Proceedings of the Institution of Mechanical Engineers, Part F: Journal of Rail and Rapid Transit* 2013; .
20. Li Y, Jiang JZ, Neild S. Inerter-based configurations for main landing gear shimmy suppression. *Journal of Aircraft* 2016; DOI: **10.2514/1.C033964**.
21. Jiang JZ, Smith MC. Regular positive-real functions and five-element network synthesis for electrical and mechanical networks. *IEEE Transactions on Automatic Control* June 2011; **56**(6):1275–1290.
22. Jiang JZ, Smith MC. Series-parallel six-element synthesis of the biquadratic impedances. *IEEE Transactions on Circuits and Systems I* November 2012; **59**(11):2543–2554.
23. Chen MZQ, Wang K, Shu Z, Li C. Realizations of a special class of admittances with strictly lower complexity than canonical forms. *IEEE Transactions on Circuits and Systems I: regular papers* September 2013; **60**(9):2465–2473.
24. Wang K, Chen MZQ, Hu Y. Synthesis of biquadratic impedances with at most four passive elements. *Journal of Franklin Institute* 2014; **351**:1251–1267.
25. Hughes TH, Smith MC. On the minimality and uniqueness of the Bott-Duffin realisation procedure. *IEEE Transactions on Automatic Control* 2014; **59**(7):1858–1873.
26. Jiang JZ, Smith MC. Regular positive-real functions and passive networks comprising two reactive elements. European Control Conference (ECC), 2009; 219–224.
27. Dittmer G. Zur realisierung von RLC-Brückenzweipolen mit zwei Reaktanzen und mehr als drei Widerständen (on the realisation of RLC two-terminal bridge networks with two reactive and more than three resistive elements). *Nachrichtentechnische Zeitschrift* 1970; **23**:225–230.
28. Foster RM, Ladenheim EL. A class of biquadratic impedances. *IEEE Transactions on Circuit Theory* 1963; :262–265.
29. Valkenburg MEV. *Introduction to Modern Network Synthesis*. John Wiley & Sons, 1960.
30. Chen MZQ, Smith MC. A note on tests for positive-real functions. *IEEE Transactions on Automatic Control* 2009; **54**(2):390–393.
31. Lin PM. A theorem on equivalent one-port networks. *IEEE Transactions on Circuit Theory* Dec 1965; **CT-12**:619–621.